ELSEVIER

Journal of Pure and Applied Algebra 120 (1997) 51–65

JOURNAL OF
PURE AND
APPLIED ALGEBRA

t-valuations and the theory of quasi-divisors

Jiří Močkoř *

Department of Mathematics, University of Ostrava, CZ-702 00 Ostrava, Czech Republic

Communicated by C.A. Weibel; received 12 August 1994; revised 28 April 1995

Abstract

Various types of *po*-groups with a quasi-divisor theory (including *po*-groups with an independent theory of quasi-divisors and *po*-groups with a theory of quasi-divisors of finite character) are characterized by various t-valuations. Relationships between *po*-groups and integral domains (including relationships between *po*-groups with independent theory of quasi-divisors and independent rings of Krull type and relationships between well behaved integral domains and their corresponding groups of divisibility) are investigated. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 13F05, 06F15

1. Introduction

Aubert [2] introduced the notion of a theory of quasi-divisors for directed partially ordered groups (*po*-groups) as a natural generalization of the theory of divisors which was introduced by Skula [21]. It is well known that there exists an internal characterization of *po*-groups which admit a theory of divisors, namely, positive cones of these groups are *Krull monoids* (see [5]). An internal characterization of *po*-groups with a theory of quasi-divisors is also available: These groups are t-Prüfer groups (i.e., every finitely generated t-ideal is t-invertible), as was proved by Jaffard [14]. A very important role in the characterization of *po*-groups with a theory of divisors is *valuations* of these groups, i.e., order homomorphisms of these groups onto totally ordered groups *po*-groups. Using these valuations, it is possible to characterize a group with a theory of divisors as a *po*-group G with a family W of valuations of G onto o-groups of integers $\mathbb{Z}$ such that for each $g \in G$, $w(g) = 0$ for almost all $w \in W$ (see [5]). In [8] it was proved that an analogous characterization is also valid for *po*-groups with a quasi-divisors theory of finite character.

* E-mail: jiri.mockor@osu.cz. Supported in part by the Grant Agency of the Czech Republic (201/93/2122).

0022-4049/97/$17.00
PII S0022-4049(96)00059-X

In this paper we would like to characterize various types of *po*-groups with a quasi-divisor theory by various t-valuations. One of the principal results of this paper is the relationship between *po*-groups with an independent theory of quasi-divisors which were introduced in [8] and independent rings of Krull type which were introduced by Griffin [10]. Moreover, we show that the notion of a well-behaved domain introduced by Zafrullah [23] is of purely multiplicative character.

All results in this paper concern *po*-groups. But all these results can be simply modified for commutative and cancellative reduced monoids, since if S is such a monoid and $Q(S)$ is the total quotient group of S, then $Q(S)$ is a *po*-group ordered by the division relation with respect to S, i.e.,

$$a, b \in Q(S), a \leq b \Leftrightarrow (\exists s \in S)\ b = as.$$

Then $Q(S)_+ = S$ and all results for a *po*-group $Q(S)$ may in a natural way be modified for S. For more relationships between *po*-groups and semigroups, also see [8].

The notion of a *po*-group with a theory of quasi-divisors was explicitly introduced for the first time by Aubert. P. Jaffard [14] proved several characterizations of these groups without mentioning the relationship between these groups and groups with a theory of divisors. Recall that a directed *po*-group $(G, ., \leq)$ has a *theory of quasi-divisors* if there exists a lattice ordered group (l-group) $(\Gamma, ., \wedge, \vee)$ and a map $h : G \to \Gamma$ such that

(1) h is an order isomorphism from G into Γ,

(2) $(\forall \alpha \in \Gamma)(\exists g_1, \ldots, g_n \in G)\ \alpha = h(g_1) \wedge \cdots \wedge h(g_n)$.

If Γ is a free abelian group $\mathbb{Z}^{(P)}$ with componentwise ordering, then h is called a *theory of divisors.*

t-ideals are an important tool for investigating groups with a theory of quasi-divisors. Recall that a t-ideal generated by a finite set X in a directed *po*-group G is the set $X_t = \{g \in G : \text{if } s \in G \text{ is a lower bound of } X, \text{ then } g \geq s\}$. If X is a lower bounded subset of G, then $X_t = \bigcup\{K_t : K \text{ is a finite subset in } X\}$. We may define on the set of t-ideals $\mathscr{I}_t(G)$ a t-multiplication $X_t \times_t Y_t = (X_t Y_t)_t = (XY)_t$. If $h : G \to \Gamma$ is a theory of quasi-divisors, then Γ is o-isomorphic to the *Lorenzen t-group of* G, i.e., the group of all finitely generated t-ideals of G under t-multiplication. Further, we say that a group homomorphism $f : G_1 \to G_2$ is a *t-homomorphism* if $f(X_t) \subseteq (f(X))_t$ for any lower bounded subset $X \subseteq G_1$. A t-homomorphism is then called a *t-valuation* if G_2 is a totally ordered group, i.e., an o-group. For more details about t-systems, see e.g. [2, 14].

2. Theory of quasi-divisors of finite character

In the valuation theory of commutative fields, the notion of an essential valuation is well known. Recall that a valuation w of a field K with valuation ring R_w and value group G_w is *essential* on an integral domain A with quotient field K if $R_w = A_{P(w)}$, where $P(w) = \{a \in A\colon w(a) > 0\}$. The notion of an essential valuation was also introduced for valuations of a monoid S (a monoid here will always be commutative, cancellative, and reduced) as a homomorphism $w : Q(S) \to G_w$ such that $w(S) \subseteq G_w^+$ and for any $z \in Q(S)$ such that $w(z) \geq 1$ there exists $x \in S$ such that $w(x) = 1$ and $zx \in S$ (see [8]). To unify both these definitions we say that an *essential t-valuation* of a directed *po*-group G is a t-homomorphism w of G onto an o-group G_w such that ker w is a directed subgroup of G_w, i.e., ker w is an *o-ideal* of G_w and w is an o-epimorphism.

In [8, Theorem 3.5], we proved that for any *po*-group G with a theory of quasi-divisors $h : G \to \Gamma$ there exists a defining family of essential t-valuations. Recall that a family W of t-valuations is called a *defining family for G*, if

$$(\forall g \in G)\ g \geq 1 \Leftrightarrow (\forall w \in W)\ w(g) \geq 1.$$

We say that W is of finite character, if

$$(\forall g \in G)(\forall' w \in W)\ w(g) = 1,$$

where $\forall'$ means "for all but a finite number". Then a theory of quasi-divisors of G is said to be of *finite character*, if there exists a defining family of t-valuations of finite character for G. In [8, Theorem 3.8], we presented some characterizations of such *po*-groups and in [8, Theorem 5.5], we proved that an integral domain R is a ring of Krull type if and only if its group of divisibility $G(R)$ has a theory of quasi-divisors of finite character. Recall that an integral domain R is a *ring of Krull type* if it is defined by a family of essential valuations of finite character. Moreover, the *group of divisibility* of an integral domain R is the multiplicative group $G(R) = K^\times/U(R)$, where $K^\times = K \setminus \{0\}$ is the multiplicative group of the quotient field K of R and $U(R)$ is the group of units of R.

Griffin [11] proved another characterization of rings of Krull type by using the so called Conrad's (F)-conditions. We prove an analogous characterization for *po*-groups with a theory of quasi-divisors of finite character.

First recall that a lattice ordered group (i.e., an l-group) Γ satisfies *Conrad's (F)-conditions* if each positive element of Γ is greater than only a finite number of pairwise disjoint elements.

Theorem 2.1. *Let G be a directed po-group. Then the following statements are equivalent:*

(1) *There exists a theory of quasi-divisors $h : G \to \Gamma$ of G such that Γ satisfies Conrad's (F)-conditions.*

(2) *G is a t-Prüfer po-group in which no element belongs to an infinite number of maximal t-ideals.*

(3) *G is defined by a family of essential t-valuations of finite character.*

(4) *G admits a theory of quasi-divisors of finite character.*

Proof. (1) ⇒ (2): Let $h : G \to \Gamma$ be a theory of quasi-divisors such that Γ satisfies Conrad's (F)-conditions. It follows that G is a t-Prüfer *po*-group (see [14, 2]). Let $g \in G$. Then any set $A \subseteq \Gamma$ of pairwise disjoint elements of Γ such that $h(g) \geq \alpha$ for all $\alpha \in A$ is finite. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a maximal subset with this property. Since h is a theory of quasi-divisors, for each $i, 1 \le i \le n$, there exists a finite subset $A^i \subseteq G$ such that $\alpha_i = \inf h(A^i)$. Hence, $h(g) \in (h(A^i))_t$ for each i, and it follows that also $g \in A^i_t$. Let P_i be a maximal t-ideal of G such that $A^i \subseteq P_i$. The existence of P_i follows from [14, Ch. I, par. 4, Theorem 9]. Then $g \in P_i$ for any i, and there is no other maximal t-ideal containing g. In fact, let $P \subseteq G$ be a maximal t-ideal such that $g \in P$ and $P \ne P_i$ for each i. Then for each i there exist $b_i \in P \setminus P_i$ and $a_i \in P_i \setminus P$. We then have $A^i_t \nsubseteq P$ for all i. In fact, if for some i, $A^i_t \subseteq P$, then $a_i \notin A^i_t$ and $b_i \notin A^i_t$, and we have

$$(A^i_t, a_i)_t \nsubseteq (A^i_t, b_i)_t \quad \text{and} \quad (A^i_t, b_i)_t \nsubseteq (A^i_t, a_i)_t.$$

Let $\beta_1 = \inf h((A^i_t, a_i)_t)$, $\beta_2 = \inf h((A^i_t, b_i)_t)$. Then $h(g) \ge \alpha_i = \inf h(A^i_t) \ge \beta_1, \beta_2$ and $\beta_1 \not\ge \beta_2$ and $\beta_2 \not\ge \beta_1$. Set $\beta'_1 = \beta_1(\beta_1 \wedge \beta_2)^{-1}$ and $\beta'_2 = \beta_2(\beta_1 \wedge \beta_2)^{-1}$. Then $\beta'_1 \wedge \beta'_2 = 1$ and $h(g) \ge \beta_i \ge \beta'_i$. But in this case,

$$\{\alpha_1, \ldots, \alpha_{i-1}, \beta'_1, \beta'_2, \alpha_{i+1}, \ldots, \alpha_n\}$$

is a larger set than the set $\{\alpha_1, \ldots, \alpha_n\}$, a contradiction.

Therefore, $A^i_t \nsubseteq P$ for all i. But in this case, it follows that $\bigcap_i A^i_t \nsubseteq P$, since otherwise $\prod_i A^i_t \subseteq \bigcap_i A^i_t \subset P$, which contradicts the fact that P is a prime t-ideal. It follows that $\alpha_1 \vee \cdots \vee \alpha_n \ge \alpha_{n+1} = \inf h(P)$ and $\alpha_{n+1} \not\ge \alpha_i$ for all i. Then set

$$\alpha'_{n+1} = \alpha_{n+1}(\alpha_{n+1} \wedge (\alpha_1 \vee \cdots \vee \alpha_n))^{-1},$$
$$\alpha'_i = \alpha_i(\alpha_i \wedge \alpha_{n+1})^{-1}, \quad i = 1, \ldots, n.$$

According to [11, Lemma 6], $\{\alpha'_1, \ldots, \alpha'_{n+1}\}$ is a set of pairwise disjoint elements such that $\alpha'_i \le \alpha_i \le h(g)$ for all i, which contradicts the maximality of the original set. Therefore, (2) holds.

(2)⇒(3): Let $\mathcal{M}_G$ be the set of all maximal t-ideals of G and let $\mathcal{P}_G$ be the set of all prime t-ideals of G. Then it follows from [17, 2.9], that the canonical map $G \xrightarrow{w_P} G/[P]$ is a t-valuation for all $P \in \mathcal{P}_G$, where $[P]$ is the convex subgroup of G generated by $G_+ \setminus P$. Moreover, we have $\bigcap_{M \in \mathcal{M}_G}[M] = \bigcap_{P \in \mathcal{P}_G}[P] = \{1\}$. Hence, $W = \{w_M : M \in \mathcal{M}_G\}$ is a defining family for G of finite character, and since $[M]$ is an o-ideal, w_M is an essential t-valuation.

(3)⇒(4): This was proved in [8, 3.8].

(4)⇒ (1): Let $h : G \to \Gamma$ be a theory of quasi-divisors of finite character, i.e., there exists a defining family W of t-valuations for G of finite character. Let $\widehat{W}$ be the set of canonical extensions of valuations from W onto valuations (i.e., l-homomorphisms) of Γ. According to [8, 3.4], $\widehat{W}$ is a defining family of Γ of finite character. Let $\alpha \in \Gamma_+$ and let $I \subseteq \Gamma_+$ be a subset of pairwise disjoint elements such that $\alpha \ge \beta$ for all $\beta \in I$. For $\beta \in I \cup \{\alpha\}$ we set $W_\beta = \{\hat{w} \in \widehat{W} : \hat{w}(\beta) \ne 1\}$. Then $W_\beta \subseteq W_\alpha$ for all $\beta \in I$, and W_α and W_β are finite. Assume that I is not finite. Then there exist $\beta, \gamma \in I, \beta \ne \gamma$, such that $W_\beta = W_\gamma$. But in this case we have $\beta \wedge \gamma \ne 1$, since for $\hat{w} \in W_\beta = W_\gamma$ we have $\hat{w}(\beta) \wedge \hat{w}(\gamma) > 1$, a contradiction. Hence, Γ satisfies Conrad's (F)-conditions. □

In [18] we presented a method for constructing *po*-groups with a strong theory of quasi-divisors as special subgroups of the restricted Hahn group $H(\Delta,\mathbb{Z})$ on a root system Δ. Recall that $H(\Delta,\mathbb{Z})$ is the group $\mathbb{Z}^{(\Delta)}$ such that

$$a \in H(\Delta,\mathbb{Z}),\ a \geq 0 \quad \Leftrightarrow \quad a_\alpha > 0 \text{ for all } \alpha \in \mathrm{ms}(a),$$

where $\mathrm{ms}(a)$ is the set of all maximal elements in $\mathrm{supp}(a)$. Recall that a partially ordered set Δ is *finitely atomic* if for any $\alpha \in \Delta$ the set $\{\sigma : \sigma \text{ is an atom in } \Delta, \sigma \leq \alpha\}$ is nonempty and finite. In [18, 3.1], we proved that if Δ is finitely atomic, then $H(\Delta,\mathbb{Z})$ is finitely atomic as well.

Proposition 2.2. *If Δ is finitely atomic, then $H(\Delta,\mathbb{Z})$ satisfies the Conrad's (F)-conditions.*

Proof. Let $a \in H(\Delta,\mathbb{Z})$, $a > 0$ and suppose that there exists an infinite set I in $H(\Delta,\mathbb{Z})$ of pairwise disjoint elements smaller than a. According to [18], $H(\Delta,\mathbb{Z})$ is finitely atomic, and it follows that for any $b_1, b_2 \in H(\Delta,\mathbb{Z})_+$ such that $b_1 \wedge b_2 = 0$, $b_i \leq a$, the sets $A(b_i) = \{s : s \text{ is an atom in } H(\Delta,\mathbb{Z}), s \leq b_i\}$, $i = 1,2$, are finite, and $A(b_1) \cap A(b_2) = \emptyset$. Hence, we have

$$\bigcup_{b\in I} A(b) \subseteq \{s : s \text{ is atom in } H(\Delta,\mathbb{Z}), s \leq a\} (= A),$$

and A is infinite, a contradiction. □

In [18] we introduced a method for constructing subgroups G of $H(\Delta,\mathbb{Z})$ for which the inclusion map $G \to H(\Delta,\mathbb{Z})$ is a theory of quasi-divisors. From Theorem 2.1 and Proposition 2.2 it follows that if Δ is finitely atomic, all such subgroups admit a theory of quasi-divisors of finite character.

Griffin [10] introduced the notion of an *independent ring of Krull type* as a ring of Krull type which is defined by a family of pairwise independent valuations. In [8] a *po*-group G with an *independent theory of quasi-divisors* was analogously introduced as a *po*-group with a theory of quasi-divisors for which there exists a defining family W of t-valuations such that elements of W are pairwise independent. Here we recall that if w_1, w_2 are two t-valuations of a *po*-group G, then w_1 is said to be coarser than w_2 ($w_1 \geq w_2$) if there exists an o-epimorphism d_{w_1,w_2} such that $w_2 = d_{w_1,w_2} w_1$. It may be then proved (see [17]) that for any two t-valuations w_1, w_2 there exists the infimum $w_1 \wedge w_2$ (with respect to this preorder relation). Then $d_{w_1w_2}$ (resp., $d_{w_2w_1}$) is an o-epimorphism such that $w_1 \wedge w_2 = d_{w_1w_2} w_1 = d_{w_2w_1} w_2$. Hence, $d_{w_1w_2} = d_{w_1,w_1\wedge w_2}$. If W is a system of t-valuations of a *po*-group G and $W' \subseteq W$, then a system $(g_w)_w \in \prod_{w\in W'} G_w$ of elements is called *compatible* provided $d_{wv}(g_w) = d_{vw}(g_v)$ for all $w, v \in W'$. W' is said to be an *independent family* if for all $w, v \in W'$ with $w \neq v$, $w \wedge v$ is the trivial t-valuation. Finally, $(g_w)_{w\in W'}$ is called *complete* if $\bigcup_{w\in W'} W(g_w) = W'$, where $W(g_w) = \{v \in W : d_{wv}(g_w) \neq 1\}$.

In order to prove some characterizations of groups with an independent theory of quasi-divisors we first need some notations and a lemma.

Let W be a defining family of t-valuations of a *po*-group G. For a lower bounded subset X of G we set

$$X_r = \{g \in G: (\forall w \in W)(\exists a \in X)\ w(g) \geq w(a)\}.$$

Jaffard [14] proved that this defines an r-system of ideals of a *po*-group G (the r-system defined by W). Further, W is said to satisfy the *approximation theorem* if for all $w_1, \ldots, w_n \in W$ and any compatible and complete system $(g_1, \ldots, g_n)$, there exists $g \in G$ such that

$$w_i(g) = g_i, \quad i = 1, \ldots, n,$$

$$w(g) \geq 1, \quad w \in W \setminus \{w_1, \ldots, w_n\}.$$

Lemma 2.3. *Let W be a defining family of t-valuations of finite character for a po-group G. For $w \in W$, set $M_w = \{g \in G: w(g) > 1\}$.*

(1) The r-system defined by W is the t-system.

(2) For any t-ideal P_t of G and any $g \in G$,

$$g \in P_t \Leftrightarrow (\forall w \in W)\ w(g) \in (w(P))_t.$$

(3) If W is an independent family and satisfies the approximation theorem, then $\{M_w : w \in W\}$ is the set of all maximal prime t-ideals of G.

Proof. (1) Let X be a lower bounded subset of G and let $g \in X_t$. Then there exists a finite subset $K \subseteq X$ such that $g \in K_t$, and for any $w \in W$ we have $w(g) \in w(K_t) \subseteq (w(K))_t = (w(k_w))_t$ for some $k_w \in K$. Hence, $w(g) \geq w(k_w)$ and it follows that $g \in X_r$. Conversely, let $g \in X_r$ and let $a_1 \in X$ be arbitrary. Then there exist at most a finite number $w_2, \ldots, w_n \in W$ such that

$$1 = w(g) = w(a_1), \quad w \in W \setminus \{w_2, \ldots, w_n\}.$$

Since $g \in X_r$, for any i, $1 \leq i \leq n$, there exists $a_i \in X$ such that $w_i(g) \geq w_i(a_i)$. It follows that $g \in (a_1, a_2, \ldots, a_n)_r$. Hence, the r-system X_r is of finite character and it follows that $X_r \subseteq X_t$.

(2) Let P be an arbitrary t-ideal of G and let $g \in G$. If $g \in P_t$, for all $w \in W$ we have $w(g) \in w(P_t) \subseteq (w(P))_t$. Conversely, if $w(g) \in (w(P))_t$ for all $w \in W$, then for any w there exists a finite subset $K_w \subseteq P$ such that $w(g) \in (w(K_w))_t = (w(k_w))_t$ for some $k_w \in K_w$. Hence, $g \in P_r = P_t$ according to (1).

(3) It is clear that M_w is a t-ideal. Let $v, w \in W$, $w \neq v$. Then $(1, g_w)$ is a complete and compatible system for any $g_w \in G_w, g_w > 1$, and according to the approximation theorem there exists $g \in G_+$ such that $w(g) = g_w, v(g) = 1$. Hence, $M_w \not\subseteq M_v$ and analogously $M_v \not\subseteq M_w$. We show that M_w is maximal. Let $g \in G$ be such that $w(g) = 1$. Then $(M_w, g)_t = G_+$. In fact, let $v \in W$. If $v = w$, we have $1 \in (v(M_w), 1)_t$. If $v \neq w$, then there exists $a \in M_w \setminus M_v$ and $1 \in (v(a), v(g))_t \subseteq (v(M_w), v(g))_t$. According to (2), $(M_w, g)_t = G_+$ and M_w is a maximal t-ideal. Now let M be another maximal t-ideal

in G. According to (2), there exists $w \in W$ such that $1 \notin (w(M))_t$, and it follows that $M \subseteq M_w$. Hence, $M = M_w$. □

Theorem 2.4. *Let G be a po-group with a theory of quasi-divisors. Consider the following statements*:

(1) *G admits an independent theory of quasi-divisors.*

(2) *Any prime t-ideal of G is contained in a unique maximal t-ideal.*

Then (2) $\Rightarrow$ (1). *If G admits a theory of quasi-divisors of finite character, then also* (1) *implies* (2).

Proof. (2) $\Rightarrow$ (1): Since G is a t-Prüfer group, according to [17, 2.9], for any prime t-ideal P of G the canonical o-homomorphism $w_P : G \rightarrow G/[P]$ is a t-valuation, where $[P]$ is the directed subgroup generated by $G_+ \setminus P$ in G (i.e., a t-local o-ideal). Then $W = \{w_M : M$ is maximal t-ideal of $G\}$ is a defining family for G (see [17, Corollary and 2.9]). If for some different maximal t-ideals M_1, M_2, the t-valuations w_{M_1}, w_{M_2} are not independent, there exists a nontrivial t-valuation w of G such that $w = w_{M_1} \wedge w_{M_2}$. Then according to [2, Theorem 8], w is equivalent to some t-valuation w_P for some prime t-ideal P of G, and it follows that $P \subseteq M_1 \cap M_2$, a contradiction.

Now suppose that G admits a theory of quasi-divisors of finite character and let (1) hold. Let W be an independent defining family of t-valuations for G of finite character. For any $w \in W$, let $\hat{w}$ be an extension of w onto a t-valuation of the Lorenzen t-group $\Gamma = \Lambda_t(G)$ such that $\hat{w}h = w$, where $h : G \rightarrow \Gamma$ is the natural embedding. Then $\ker \hat{w}$ is a prime l-ideal of Γ and according to [17, 2.9], there exists a prime t-ideal P of G such that for the canonical t-valuation $w_P : G \rightarrow G/[P]$ we have $w_P = \varphi w$ for some o-isomorphism φ. Hence, there exists a set $\mathscr{P}$ of prime t-ideals of G such that $W_0 = \{w_P : P \in \mathscr{P}\}$ is an independent defining family of finite character for G. Suppose that for some $P \in \mathscr{P}$, P is not a maximal t-ideal. Let M be a maximal t-ideal of G such that $P \subseteq M$. Then w_M is a t-valuation of G and $W' = (W \setminus \{w_P\}) \cup \{w_M\}$ is an independent defining family of finite character for G. In fact, it is clear that W' is a defining family for G. If there exists $Q \in \mathscr{P}$ ($Q \neq P$) such that w_M and w_Q are not independent, then there exists a nontrivial t-valuation w of G such that $w = w_M \wedge w_Q \leq w_M$. Since $w_P < w_M$, the valuations w and w_P are comparable. If $w \geq w_P$, then $w_P \leq w \leq w_Q$, and it follows that $w_P \wedge w_Q$ is nontrivial, a contradiction. If $w < w_P$, then $w_P \wedge w_Q \geq w$, a contradiction. Hence, W' is independent.

Repeating this procedure, we obtain an independent defining family of t-valuations $W = \{w_M : M \in \mathscr{M}\}$, where $\mathscr{M}$ is a set of maximal t-ideals. We show that in this case $\mathscr{M}$ contains all maximal t-ideals of G. In fact, let M_t be a maximal t-ideal of G. Since $1 \notin M_t$ it follows from Lemma 2.3(2) that there exists $P \in \mathscr{P}$ such that $1 \notin (w_P(M))_t$. If P is maximal, we have $P \in \mathscr{M}$. Otherwise, there exists $Q \in \mathscr{M}$ such that $P \subseteq Q$ (this follows from the construction of $\mathscr{W}$). Since $w_P \leq w_Q$, we have $1 \notin (w_Q(M))_t$. Let $T_Q = \{g \in G_+ : w_Q(g) > 1\}$. It is clear that T_Q is a t-ideal in G and $M = M_t \subseteq T_Q$. Since M is a maximal t-ideal, $M = T_Q$. On the other hand, $T_Q = Q$ since w_Q is the canonical homomorphism $G \rightarrow G/[Q]$. It follows that $M = Q \in \mathscr{M}$.

Now, if there exists a prime t-ideal P of G such that $P \subseteq M_1 \cap M_2$ for different maximal prime t-ideals M_i, we have $w_{M_i} \in W$ and $w_P \le w_{M_1} \wedge w_{M_2}$, a contradiction. □

We now want to derive some relationships between *po*-groups which are defined by two defining families.

Let G and G' be *po*-groups, $h : G \to G'$ an o-homomorphism, and let W and W', respectively, be defining families of t-valuations of G and G'. Then W' is said to be *coarser than* W (*with respect to* h), in symbols $W' \le_h W$, if there exists an injective map $\sigma : W' \to W$ such that for each $w' \in W'$ there exists an o-homomorphism $h_{w'}$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & G' \\ {\scriptstyle \sigma(w')}\downarrow & & \downarrow{\scriptstyle w'} \\ G_{\sigma(w')} & \xrightarrow{h_{w'}} & G_{w'} \end{array}$$

In [18, Proposition 2.4], we proved that if G has a theory of quasi-divisors and H is an o-ideal of G, then G/H has a theory of quasi-divisors as well. Moreover, in this case there exists a defining family of t-valuations for G/H. In next propositions we want to show that there is a deeper relationship between defining families for G and G/H . First, the following simple proposition holds.

Proposition 2.5. *Let G and G' be po-groups and let $h : G \to G'$ be an o-epimorphism. Let W and W' be defining families of t-valuations of G and G', respectively, and let W' be coarser than W with respect to h. Then*

(1) *If W is of finite character, then W' is of finite character.*

(2) *If every $w \in W$ is essential on G, then every $w' \in W'$ is essential on G'.*

(3) *If W is an independent family, then W' is an independent family.*

Proof. Let $\sigma : W' \to W$ be the injection from the defininiton of $W' \le_h W$.

(1) Let $g' \in G', g' \neq 1$, and let $g \in G$ be such that $h(g) = g'$. Then if $w'(g') \neq 1$, we have $\sigma(w')(g) \neq 1$ and σ induces an injection

$$\{w' \in W' : w'(g') \neq 1\} \to \{\sigma(w') \in W : \sigma(w')(g) \neq 1\} \\ \subseteq \{w \in W : w(g) \neq 1\}.$$

Therefore, if W is of finite character, the same property holds for W'.

(2) Let $g' \in \ker w'$ and let $g \in G$ be such that $h(g) = g'$. If $w(g) \ge 1$ then there exists $t \in \ker w$ such that $gt \ge 1$. Since w is essential, there exists $t_1 \in \ker w$ such that $t_1 \ge 1$ and $t_1 \ge t$. Hence, $gt_1 \ge g$ and $gt \ge 1$ and $h(t_1) \ge g'$ and $h(t_1) \ge 1$, $h(t_1) \in \ker w$. Since $w'h = h_w \sigma(w')$ is an o-epimorphism, it follows that w' is an o-epimorphism as well.

(3) This follows easily from the definition of $W' \le_h W$ □

If (G,x) and (G_1,y) are partially ordered groups with r-systems, then a homomorphism $\varphi : G \to G_1$ is called a (x,y)-morphism if for any lower bounded subset $A \subseteq G$, $\varphi(A_x) \subseteq (\varphi(A))_y$ holds.

Lemma 2.6. *Let H be an o-ideal of a po-group G. Then the canonical o-homomorphism $\varphi : G \to G/H$ is a (t,t)-morphism.*

Proof. Let A be a lower bounded subset of G and let $x \in A_t$. Then there exists a finite subset $K \subseteq A$ such that $x \in K_t$. Let $\alpha = \varphi(a)$ be a lower bound of $\varphi(K)$ in G/H. Then for any $k \in K$ there exists $h_k \in H$ such that $a + h_k \leq k$. Since H is directed, there exists $h \in H$ such that $h \leq h_k$ for all $k \in K$, and we obtain $\varphi(x) \geq \alpha$. Hence, $\varphi(x) \in (\varphi(K))_t \subseteq (\varphi(A))_t$. □

Proposition 2.7. *Let W be a defining family of t-valuations of finite character for a po-group G and let H be an o-ideal of G. Then there exists a defining family W_H of t-valuations for G/H such that $W_H \leq_h W$, where $h : G \to G/H$ is the canonical o-epimorphism.*

Proof. Let $w \in W$. We set

$$H_w = \{v\colon v \text{ is a } t\text{-valuation of } G, H \subseteq \ker v,\ \ker w \subseteq \ker v\}.$$

Let K be the *core* of $X = \bigcap_{v \in H_w} \ker v$, i.e. the convex directed subgroup generated by the positive elements of X. Then K is an o-ideal of G, and $H \subseteq K$ and $\ker w \subseteq K$. It follows that G/K is an o-group, and according to Lemma 2.6, the canonical o-homomorphism $\bar{w} : G \to G/K$ is a t-valuation. Hence, $\bar{w}$ is the greatest element in H_w.

Now, let $w' : G/H \to G/K$ be the canonical o-epimorphism defined by $\bar{w}$, i.e., $w'h = \bar{w}$. Since $\ker w'$ is directed, w' is a t-valuation according to Lemma 2.6. Let $W_H = \{w'\colon w \in W\}$. Since there exists a map from W onto W_H, we obtain an injection $\sigma : W_H \to W$ such that the diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{h} & G/H \\ {\scriptstyle \sigma(w')}\downarrow & & \downarrow{\scriptstyle w'} \\ G_{\sigma(w')} = G_w & \longrightarrow & G/\ker \bar{w} = G_{w'} \end{array}$$

Hence, $W_H \leq_h W$. We show that W_H is a defining family for G/H. Let $gH \in (G/H)_+$. Since $H \subseteq \ker w'$, we have $w'(gH) \geq 1$ for all $w' \in W_H$. Conversely, let $xH \in G/H$ be such that $w'(xH) \geq 1$ for all $w' \in W_H$. For $a \in H_+$, put $W_a = \{w \in W\colon w(xa) < 1\}$. Since W is of finite character, W_a is finite for any a, and there exists $a \in H_+$ such that W_a is minimal in the set of all W_b, $b \in H_+$, ordered by inclusion. Assume that $W_a \neq \emptyset$ and let $w \in W_a$. Then $w(ax) < 1$, and there exists a greatest convex subgroup Δ in G_w such that $w(ax) \notin \Delta$. Let v be the t-valuation $G \xrightarrow{w} G_w \xrightarrow{\text{nat}} G_w/\Delta$. Then $v \leq w$ and $v(ax) < 1$. Consider the only two possible cases.

(1) $H \subseteq \ker v$. Since $\ker w \subseteq \ker v$, we have $v \in H_w$ and $v \leq \bar{w}$, the greatest element in H_w. Then $w' \in W_H$ is a t-valuation such that the following diagram commutes:

$$\begin{array}{ccccccc} G_w & \longleftarrow & G & = & G & \longrightarrow & G/H \\ \downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \bar{w}} & & \downarrow{\scriptstyle w'} \\ G_w/\Delta & = & G_w/\Delta & \overset{\varphi}{\longleftarrow} & G_{\bar{w}} & = & G_{\bar{w}} \end{array}$$

According to the assumption, we have $\bar{w}(xa) = w'(xH) \geq 1$. Then $v(ax) = \varphi(\bar{w}(xa)) \geq 1$, a contradiction.

(2) $H \nsubseteq \ker v$. Then there exists $b \in H$ such that $v(b) > 1$. Assume that $w(b^n ax) \leq 1$ for all natural numbers n. Let Δ' be the convex subgroup of G_w generated by $w(b)$. Then $w(ax) \notin \Delta'$, and it follows that $\Delta' \subseteq \Delta$. Therefore, $w(b) \in \Delta' \subseteq \Delta$ and $v(b) = 1$, a contradiction. Hence, there exists a natural n such that $w(b^n ax) > 1$. We have $b^n a \in H$ and $W_{b^n a} \subseteq W_a \setminus \{w\}$. In fact, $w \notin W_{b^n a}$, and for $u \in W_{b^n a}$ we have $u(b^n ax) < 1$, and it follows that $1 \geq u(b^n)^{-1} > u(ax)$. Hence, $u \in W_a$. It follows that W_a is not minimal, a contradiction. Therefore, $W_a = \emptyset$ and $xa \geq 1$. It follows that W_H is a defining family for G/H. □

Recall that a theory of quasi-divisors $h : G \to \Gamma$ is called a *strong theory of quasi-divisors* (see [17]), if

$$(\forall \alpha, \beta \in \Gamma_+)(\exists \gamma \in \Gamma_+)\ \alpha.\gamma \in h(G), \beta \wedge \gamma = 1.$$

It may be proved that any strong theory of quasi-divisors is also a theory of quasi-divisors. Moreover, if h is a strong theory of quasi-divisors and Γ is the free abelian group $\mathbb{Z}^{(P)}$ with componentwise ordering, then h is called a *strong theory of divisors* [21]. It is well known that in this case every element of Γ_+ is the infimum of two elements of $h(G)$ (see [21, 2.2]). Moreover, since the group of divisibility $G(A)$ of an integral domain A admits a (strong) theory of divisors if and only if A is a Krull domain, it follows in this case that every finitely generated t-ideal of $G(A)$ is generated by two elements. We first prove an analogous result for *po*-groups with a strong theory of quasi-divisors.

Proposition 2.8. *Let $h : G \to \Gamma$ be a strong theory of quasi-divisors. Then every finitely generated t-ideal of G is generated by two elements.*

Proof. Let $(a_1, \ldots, a_n)_t \subseteq G_+$ and let $\alpha = h(a_1) \wedge \cdots \wedge h(a_n)$. Then there exists $\gamma_1 \in \Gamma_+$ such that $\alpha\gamma_1 = h(g_1)$ for some $g_1 \in G$. Further, there exists $\gamma_2 \in \Gamma_+$ such that $\alpha\gamma_2 = h(g_2)$ and $\gamma_1 \wedge \gamma_2 = 1$. Hence, $\alpha = \alpha(\gamma_1 \wedge \gamma_2) = h(g_1) \wedge h(g_2)$, and it follows that $(a_1, \ldots, a_n)_t = (g_1, g_2)_t$. □

If $h : G \to \Gamma$ is a theory of quasi-divisors of finite character, we may prove even a stronger version of this result.

We first prove a simple lemma.

Lemma 2.9. *Let W be a family of t-valuations of a po-group G and let $(g_{i,w})_w \in \prod_{w\in W} G_w$ be a family of compatible and complete elements for $i = 1,\dots,n$. Let*

$$a_w = \sum_{i=1}^{n} g_{i,w},$$

$$b_w = \inf\{g_{i,w} : i = 1,\dots,n\},$$

and

$$c_w = \sup\{g_{i,w} : i = 1,\dots,n\}.$$

Then $(a_w)_w, (b_w)_w$, and $(c_w)_w$ are complete and compatible families.

Proof. For $w, v \in W$ we have

$$d_{wv}(a_w) = d_{wv}\left(\sum_i g_{i,w}\right) = \sum_i d_{wv}(g_{i,w})$$
$$= \sum_i d_{vw}(g_{i,v}) = d_{vw}\left(\sum_i g_{i,v}\right) = d_{vw}(a_v)$$

and it follows that $(a_w)_w$ is a compatible family. Analogous results we receive for the rest of families and for completeness. □

Proposition 2.10. *Let $h : G \to \Gamma$ be a strong theory of quasi-divisors of finite character. Then for any finitely generated t-ideal $(a_1,\dots,a_m)_t$ of G_+ and any $k, 1 \le k \le m$, there exists $g \in G_+$ such that*

$$(a_1,\dots,a_m)_t = (a_k, g)_t.$$

Proof. Let W be a defining family of finite character for G. For $w \in W$ we set

$$g_w = w(a_1) \wedge \cdots \wedge w(a_m) \ge 1.$$

Since W is of finite character, $W_1 = \{w \in W : g_w > 1\}$ is a finite set. Let $W_2 = \{w \in W : w(a_k) > g_w\}$. Similarly, W_2 is finite. Since the families $(w(a_i))_{w\in W}$ are compatible and complete for $i = 1,\dots,m$, it follows that $(g_w)_{w\in W_1\cup W_2}$ is also compatible and complete by Lemma 2.9. Now, according to [17, Theorem 3.5], the set W satisfies the approximation theorem, and it follows that there exists $g \in G$ such that

$$w(g) = g_w; \quad w \in W_1 \cup W_2,$$

$$w(g) \ge 1; \quad w \in W \setminus (W_1 \cup W_2).$$

Hence, since W is a defining family for G, we have $g \in G_+$. Then $(a_1,\dots,a_m)_t = (a_k, g)_t$. In fact, we have $g \in (a_1,\dots,a_m)_t$. Conversely, let $x \in (a_1,\dots,a_m)_t$ and let $d \le a_k, g$. Then for $w \in W_1 \cup W_2$, we have $w(x) \ge w(d)$. If $w \in W \setminus (W_1 \cup W_2)$, then $w(g) \ge 1 = g_w = w(a_k)$, and it follows that $w(d) = 1$. Hence, $w(x) \ge w(d)$ and we have $x \ge d$. Therefore, $x \in (a_k, g)_t$. □

3. Relationships between *po*-groups and domains

In the theory of divisibility many relationships between domains and their *po*-groups of divisibility are well known. One of the first significant results in this direction is due to Skula [21] who proved that a domain is a Krull domain if and only if its group of divisibility admits a theory of divisors. Another result in this direction is due to Geroldinger and Močkoř [8]. They proved that a domain is a ring of Krull type if and only if its group of divisibility admits a quasi-divisors theory of finite character.

In this section we describe relationships between arithmetical properties of integral domains and properties of their groups of divisibility.

Griffin [10] proved that in a completely integrally closed domain of Krull type all essential valuations are rank one. In this section we show that this result is of purely multiplicative character since it may be proved using only groups of divisibility.

Recall that a directed *po*-group G is *completely integrally closed* if for any element $g \in G$ such that there exists $a \in G_+$ with $ag^n \geq 1$ for all natural number n, then $g \geq 1$. It is clear that a domain R is completely integrally closed if and only if its group of divisibility $G(R)$ is completely integrally closed. The analogical version of the following proposition was proved in the monoid theory (see [6]) firstly. Although the monoid approach is very similar to partially group approach (see [8]), we present another proof of this proposition.

Proposition 3.1. *Let W be a defining family of t-valuations of a po-group G which satisfies the approximation theorem. If G is completely integrally closed then w is rank one for each $w \in W$.*

Proof. Let us suppose that there exists $w \in W$ with rank > 1. Then there exists a proper convex subgroup H of G_w. Let $\alpha \in H$ with $\alpha < 1$ and $\beta \in G_w \setminus H$ with $\beta > 1$. Since W satisfies the approximation theorem, there exists $g \in G$ such that

$$w(g) = \alpha < 1,$$
$$w'(g) \geq 1, \quad w' \in W, w' \neq w.$$

Hence, $g \notin G_+$. Using the approximation theorem, it may be easily proved that w is an o-epimorphism. Hence, there exists $a \in G_+$ such that $w(a) = \beta$. Then we have $w(ag^n) = \beta\alpha^n > 1$. In fact, if $\beta\alpha^n \leq 1$ then since $\alpha^n \in H, \alpha^n < \beta\alpha^n \leq 1$, it follows that $\beta.\alpha^n \in H$, a contradiction with $\beta \notin H$. Now, for $w' \in W, w' \neq w$, we have $w'(ag^n) \geq w'(a) \geq 1$, and it follows that $ag^n \in G_+$ for all natural number n. Since G is completely integrally closed, we obtain $g \geq 1$, a contradiction. Therefore, $w \in W$ is of rank one. □

Now, let R be a domain of Krull type and let W be a defining family of essential valuations of finite character for R. For $w \in W$ let $\hat{w} : G(R) \to G_w$ be the canonical o-homomorphism such that $ww_R = \hat{w}$, where w_R is the semivaluation associated with R. Since any $w \in W$ is an essential valuation, there exists an o-ideal H_w in $G(R)$ such that $\hat{w}$ is equivalent to the canonical o-epimorphism $G(R) \to G(R)/H_w$. According to Lemma 2.6, $\hat{w}$ is a t-valuation. It is clear that $\widehat{W} = \{\hat{w}: w \in W\}$ is a defining family of t-valuations of finite character for $G(R)$.

Skula [21] proved that if a po-group G is the group of divisibility of an integral domain R and if G admits a theory of divisors, then it admits a strong theory of divisors. In the monoid theory an analogical result was proved for a monoid analogy of partially ordered groups with quasi divisor theory in [7]. Although this monoid theory approach is very similar to partially groups approach (see [8]), we present this result directly by using partially ordered group language.

Proposition 3.2. *Let G be a po-group which is the group of divisibility of an integral domain. If G admits a theory of quasi-divisors of a finite character, it admits a strong theory of quasi-divisors.*

Proof. Let R be an integral domain such that $G(R) = G$. According to [8, 5.4], R is a ring of Krull type and there exists a defining family W of essential valuations of finite character for R. Let $\widehat{W} = \{\hat{w}: w \in W\}$ be a corresponding defining family of t-valuations of finite character for G. According to [10, Theorem 9], the set W satisfies the approximation theorem. We show that $\widehat{W}$ satisfies the approximation theorem as well. Let $\hat{w}_1, \dots, \hat{w}_n \in \widehat{W}$ and let $(g_i)_i \in G_{\hat{w}_1} \times \cdots \times G_{\hat{w}_n}$ be a complete, compatible system with respect to $\widehat{W}$. Since any $\hat{w}$ is a t-valuation, we have $\hat{w}_i \wedge \hat{w}_j = \widehat{w_i \wedge w_j}$ and it follows that $(g_i)_i$ is also a complete and compatible system with respect to W. Hence, there exists $a \in K$ (the quotient field of R) such that

$$w_i(a) = g_i, \quad i = 1, \dots, n,$$

$$w(a) \geq 1, \quad w \in W \setminus \{w_1, \dots, w_n\}.$$

Therefore, the element $g = w_R(a)$ satisfies the conditions of the approximation theorem for $\widehat{W}$. Now, according to [17, 3.5], G admits a strong theory of quasi-divisors. □

Using the above two propositions, Griffin's result [10, Proposition 21], may be proved immediately.

We could next to investigate the relationship between the so-called well-behaved domains (introduced by Zafrullah [23]) and the corresponding groups of divisibility. An integral domain R is called *well behaved* if for any prime t-ideal P of R, PR_P is also a t-ideal. To show that this notion is of purely multiplicative character, we need to first recall some properties of t-local o-ideals (see [17]). Let H be an o-ideal of a directed po-group G and let $\varphi : G \to G/H$ be the canonical o-homomorphism. Then for any lower bounded subset $\mathscr{A} \subseteq G/H$, we may find a lower bounded subset $A \subseteq G$ such that $\{aH: a \in A\} = \mathscr{A}$. Then we set $\mathscr{A}_{t_H} = A_t/H$. In [17, 2.1], it is proved that t_H is an r-system of finite character in G/H. Then H is called a *t-local o-ideal* provided that t_H is a local r-system, i.e., in $(G/H)_+$ there exists a unique maximal t_H-ideal. In [17, Proposition 2.4], it was proved that H is a t-local o-ideal if and only if $G_+ \setminus H$ is a prime t-ideal in G.

We first list some further properties of the t_H-system. Let $\mathscr{H}_T(G,x)$ denote the set of x-local o-ideals of G containing the o-ideal T, where x is an r-system on G. Then $\mathscr{H}(G,x) := \mathscr{H}_{\{1\}}(G,x)$.

Lemma 3.3. *Let G be a directed po-group and let H be an o-ideal of G.*

(1) *$t_H \leq t$ in G/H, i.e. $X_{t_H} \subseteq X_t$ for any lower bounded subset X of G/H.*

(2) *$\mathscr{H}(G/H,t) \subseteq \mathscr{H}(G/H,t_H)$.*

(3) *There exists a bijection $\varphi : \mathscr{H}(G/H,t_H) \to \mathscr{H}_H(G,t)$ such that for any $\mathscr{T} \in \mathscr{H}(G/H,t_H)$ the canonical isomorphism $\psi : (G/H)/\mathscr{T} \to G/\varphi(\mathscr{T})$ is a $((t_H)_{\mathscr{T}}, t_{\varphi(\mathscr{T})})$-isomorphism.*

Proof. (1) Let $\mathscr{A} \subseteq G/H$ be a lower bounded subset in G/H, and let $\boldsymbol{x} \in \mathscr{A}_{t_H}$. Then there exists a lower bounded subset A in G such that $A/H = \mathscr{A}$ and $\mathscr{A}_{t_H} = A_t/H$. Hence there exists a finite subset $K \subseteq A$ and $x \in K_t$ such that $\boldsymbol{x} = xH$. We show that $\boldsymbol{x} \in (K/H)_t$. In fact, let $\alpha = aH \leq K/H$ in G/H. Then for any $k \in K$ there exists $h_k \in H$ with $ah_k \leq k$. Since H is directed, we may find $h \in H$ such that $ah \leq ah_k \leq k$, and it follows that $x \geq ah$. Therefore, $\boldsymbol{x} \in (K/H)_t \subseteq \mathscr{A}_t$.

(2) Let $\mathscr{T} \in \mathscr{H}(G/H,t)$. According to [17, 2.4], the set $\mathscr{P} = (G/H)_+ \setminus \mathscr{T}$ is a prime t-ideal in G/H. According to (1), $\mathscr{P}$ is a t_H-ideal as well, and it follows that $\mathscr{T}$ is a t_H-local o-ideal.

(3) The existence of a bijection φ follows from [17, 2.2]. The equality $\psi(\mathscr{A}_{(t_H)_{\mathscr{T}}}) = (\psi(\mathscr{A}))_{t_{\varphi(\mathscr{T})}}$ may be proved by simply computing corresponding ideals. □

We call a directed *po*-group G *well behaved* provided that for any t-local o-ideal H of G, the unique maximal t_H-ideal of G/H is a t-ideal.

Theorem 3.4. *Let R be an integral domain and let $G = G(R)$ be its group of divisibility. Then the following statements are equivalent*:

(1) *R is well behaved.*

(2) *$G(R)$ is well behaved.*

Proof. (1) ⇒ (2): Let H be a t-local o-ideal of G and let $\mathscr{P}$ be the unique maximal t_H-ideal of G/H. According to [17, 2.1], there exists a prime t-ideal Q_t of G such that $\mathscr{P} = Q_t/H$. Moreover, $\mathscr{A} = G_+ \setminus H$ is a prime t-ideal and $\mathscr{P} = \mathscr{A}/H$. Now, according to [8, 4.7], there exists a prime t-ideal P of R such that $w(P^\times) = \mathscr{A}$, where w is the canonical semivaluation of R. Since R is well behaved, PR_P is a t-ideal. But the group of divisibility of $G(R_P)$ is o-isomorphic to G/H, and $w_P = \operatorname{nat} w$ is the semivaluation associated with R_P. Hence, in G/H there exists a prime t-ideal $\mathscr{T}_t$ corresponding to PR_P, i.e., $w_P(PR_P^\times) = \mathscr{T}_t$. In this case we have $w_P(PR_P^\times) = \mathscr{A}/H = \mathscr{P}$, and it follows that $\mathscr{P}$ is a prime t-ideal of G/H.

(2) ⇒ (1): Let G be well behaved and let P be a prime t-ideal of R. Then $Q_t = w(P^\times)$ is a prime t-ideal of G and the o-ideal H generated by $G_+ \setminus Q_t$ is t-local

by [17, 2.4]. Since G is well behaved, Q_t/H is a t-ideal, and according to [8, 4.7], $PR_P = w^{-1}(Q_t/H) \cup \{0\}$ is a t-ideal. □

References

[1] M. Anderson and T. Feil, Lattice-Ordered Groups (Reidel, Dordrecht, Tokyo, 1988).

[2] K.E. Aubert, Divisors of finite character, Ann. Matem. Pura Appl. 33 (1983) 327–361.

[3] K.E. Aubert, Localizations dans les systemes d'idéaux, C.R. Acad. Sci. Paris 272 (1971) 465–468.

[4] Borevič-Šafarevič, Number Theory (Academic Press, New York, 1966).

[5] L.G. Chouinard, Krull semigroups and divisor class groups, Can. J. Math. 33 (1981) 1459–1468.

[6] A. Geroldinger, The complete integral closure of monoids and domains, PU.M.A. 4 (1993) 147–165.

[7] A. Geroldinger and F. Halter-Koch, Arithmetical theory of monoid homomorphisms, Semigroup Forum 48 (1994) 333–362.

[8] A. Geroldinger and J. Močkoř, Quasi divisor theories and generalizations of Krull domains, J. Pure Appl. Algebra 102 (1995) 289–311.

[9] R. Gilmer, Multiplicative Ideal Theory (Marcel Dekker, New York, 1972).

[10] M. Griffin, Some results on v-multiplication rings, Can. J. Math. 19 (1967) 710–722.

[11] M. Griffin, Rings of Krull type, J. Reine Angew. Math. 229 (1968) 1–27.

[12] F. Halter-Koch, Ein Approximationssatz für Halbgruppen mit Divisorentheorie, Results Math. 19 (1991) 74–82.

[13] F. Halter-Koch and A. Geroldinger, Realization theorems for semigroups with divisor theory, Semigroup Forum 201 (1991) 1–9.

[14] P. Jaffard, Les Systemes d'Idéaux (Dunod, Paris, 1960).

[15] J. Močkoř, Groups of Divisibility (Reidel, Dordrecht, 1983).

[16] J. Močkoř and J. Alajbegovic, Approximation Theorems in Commutative Algebra (Kluwer, Dordrecht, 1992).

[17] J. Močkoř and A. Kontolatou, Quasi-divisors theory of partly ordered groups, Grazer Math. Ber. 318 (1992) 81–98.

[18] J. Močkoř and A. Kontolatou, Divisor class groups of ordered subgroups, Acta Math. Inf. Univ. Ostrav. 1 (1993).

[19] J. Močkoř and A. Kontolatou, Groups with a quasi divisor theory, Comm. Math. Univ. St. Pauli Tokyo 42 (1993) 23–36.

[20] J. Ohm, Semi-valuations and groups of divisibility, Can. J. Math. 21 (1969) 576–591.

[21] L. Skula, Divisorentheorie einer Halbgruppe, Math. Z. 114 (1970) 113–120.

[22] L. Skula, On c-semigroups, Acta Arith. 31 (1976) 247–257.

[23] M. Zafrullah, Well behaved prime t-ideals, J. Pure Appl. Algebra 65 (1990) 199–207.